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Some theorems on free algebras and on direct products of algebras

W. Peremans

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SOME THEOREMS ON FREE ALGEBRAS AND ON DIRECT PRODUCTS OF ALGEBRAS

by W. Peremans (Amsterdam)

We give first the definitions of some concepts of abstract algebra, following G. Birkhoff, Lattice theory, 2nd edition, foreword on algebra.

An *algebra* A is a set of elements together with a set V of operations. These operations are functions $O(x_1, \ldots, x_n)$ which assign to every n-tuple $(a_1, \ldots, a_n)$ of elements of A an element of A. The number of variables in an operation is called the *order* of the operation. An operation of order n is also called an *n-ary* operation (for $n = 1$ a *unary*, for $n = 2$ a *binary* operation etc.). This order is assumed to be finite: the operations are *finitary*. The orders of the operations of V may be different. The cardinal number of V needs not be finite. If we wish to denote the set of operations which is assigned to the algebra A, we say that A is a V-algebra.

A subset B of an algebra A is called a *subalgebra* of A if B also is an algebra with respect to the same operations as A. This means, that if $O(x_1, \ldots, x_n)$ is an arbitrary operation and $b_1, \ldots, b_n$ are elements of B, $O(b_1, \ldots, b_n)$ is also an element of B. Obviously A is a subalgebra of itself.

If A and B are V-algebras, and there exists a mapping $a \to b$ of the elements of A on elements of B such that, if $O(x_1, \ldots, x_n)$ is an operation of V, if $a_1, \ldots, a_n$ are elements of A and if $a_i \to b_i$ in the mapping, it follows that $O(a_1, \ldots, a_n) \to O(b_1, \ldots, b_n)$ in the mapping, this mapping is called a *homomorphism* of A *into* B. If moreover for every element b of B there exists an element a of A such that $a \to b$ in the mapping, the mapping is called a homomorphism of A *onto* B. In the latter case A is called *homomorphic* to B. If a homomorphism is a one-to-one mapping, it is called an *isomorphism*. If there exists an isomorphism of A onto B (and then also an isomorphism of B onto A, viz. the inverse mapping) A and B are called *isomorphic*.

Obviously the intersection of an arbitrary non-void class of subalgebras of an algebra is also a subalgebra of that algebra. If S is an arbitrary subset of an algebra A the intersection of all subalgebras of A which contain all elements of S is called the *subalgebra of* A *generated by* S.

If A and B are V-algebras the *direct product* (*direct union*) of

A and B is defined as the set of all couples $[a, b]$ with a in A and b in B. If $O(x_1, \ldots, x_n)$ is an operation of V, we define

$$O([a_1, b_1], \ldots, [a_n, b_n]) = [O(a_1, \ldots, a_n), O(b_1, \ldots, b_n)].$$

Similarly we define the direct product of more than two factors.

For a set V of operations, a set W of symbols called *variables*, and an integer $n \geqslant 0$ we define a *V-W-polynomial* of order n by induction as follows. For every operation of V take an operation symbol with k variable-places („Leerstellen") if the original operation was k-ary. The variables are V-W-polynomials of order 0. For $n > 0$ a V-W-polynomial of order n is an operation symbol with on every variable-place substituted a V-W-polynomial of order $< n$ and on at least one place a V-W-polynomial of order $n - 1$. If α and β are V-W-polynomials, we call $\alpha = \beta$ a *V-W-identity*.

If we substitute for the variables occuring in a V-W-polynomial arbitrary elements of a V-algebra A and for the operation symbols the corresponding operations we get an element of A. The exact definition of this process, which we call *specialization*, is easily given by induction to the order of the polynomial. If the elements of A are chosen from a certain subset S of A, we speak of an S-specialization. If we apply specialization to different polynomials at once, as we shall do in the following, we must take care to substitute for the same variable in different polynomials the same element.

If the V-algebra A is homomorphic to the V-algebra B, if an A-specialization applied to a V-W-polynomial gives an element a of A and if the B-specialization with the corresponding (with respect to the homomorphism) elements of B applied to the same V-W-polynomial gives an element b of B, then b corresponds to a in the homomorphism. This fact is easily proved by induction to the order of the polynomial.

If S is a subset of a V-algebra A, the subalgebra of A generated by S consists of those and only those elements of A which are obtained by S-specialization applied to V-W-polynomials, if W is a denumerably infinite set. (If S is finite, a set of the same cardinal number as S suffices.) We omit the proof, which is completely analogous to the well-known corresponding theorem in group theory. It is essential here that all operations are finitary.

We call a *V-W-axiom* an expression of propositional calculus in which V-W-identities are substituted for the propositional variables. A set of V-W-axioms is called a *V-W-axiom system*. For the operations of propositional calculus we use the following signs: $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication and $\neg$ for negation. A V-algebra A is said to *satisfy* a V-W-axiom P if for every A-specialization applied to the V-W-polynomials occuring in P a statement about elements of A is obtained which is true. A V-algebra is said to satisfy a V-W-axiom system Q if it satisfies all axioms of Q. Such a V-algebra is called a Q-V-algebra. If there are no Q-V-algebras, Q is called *inconsistent*, if there are only Q-V algebras with one element, Q is called *semi-consistent*, if Q is neither inconsistent nor semi-consistent Q is called *consistent*.

We call a Q-V-algebra A a *free* Q-V-algebra with n generators (n a finite or infinite cardinal number) if there exists a subset S of A such that S has cardinal number n, A is generated by S and every mapping of S into an arbitrary Q-V-algebra B can be extended to a homomorphism of A into B.

This characterization of a free algebra is in accordance with that of Birkhoff, except for the fact that Birkhoff only postulates that the free algebra is a V-algebra in stead of a Q-V-algebra. The existence of the free algebra then always is guaranteed, but uniqueness does not always hold (cf. my paper ,,Een opmerking over vrije algebra's'', Math. Centrum Rapport ZW 1949—015). It seems to me that my characterization is more natural. As we shall see in the following, we now find always uniqueness, but not always existence of the free algebra.

Theorem 1. Two free Q-V-algebras with n generators are isomorphic.

Proof: We take a set W_n of cardinal number n and form the set of all V-W_n-polynomials. We call two V-W_n-polynomials equivalent if for all Q-V-algebras A every A-specialization gives for both V-W_n-polynomials the same element. Obviously this equivalence is reflexive, symmetric and transitive. We take the set B, whose elements are the classes of equivalent V-W_n-polynomials and make B a V-algebra as follows. To define the result of the operation O of V applied to a set of elements of B we choose representants from the classes and substitute them in the operation symbol corresponding to O; the resulting V-W_n-polynomial lies in a class which we take as the result of O applied to those classes. Obviously this result is independent of the choice of the representants. The V-algebra B is called the *Q-quasi-free* V-algebra with n generators. Now we prove that an arbitrary free Q-V-algebra C with n generators is isomorphic to B, from which the theorem follows. Let S

be the generating set of C and take an arbitrary one-to-one mapping between S and W_n. This mapping gives rise to an S-specialization for every V-W_n-polynomial and as C is generated by S all elements of C are obtained in this way. Moreover two equivalent V-W_n-polynomials are mapped into the same element of C, because C is a Q-V-algebra. Thus a mapping of B onto C is induced, which obviously is a homomorphism. In order to prove that it is an isomorphism, take a V-W_n-polynomial α and let its image in the S-specialization be c. Now take an arbitrary A-specialization into an arbitrary Q-V-algebra A and take the corresponding mapping of S into A. Because C is a free algebra the latter mapping may be extended to a homomorphism of C into A; let $c \to a$ in this homomorphism. The S-specialization turns α into c, the A-specialization turns W_n into the elements of A corresponding to those of S in the homomorphism, so the A-specialization turns α into a. This holds for every V-W_n-polynomial which is mapped into c, so all these V-W_n-polynomials are equivalent and the mapping of B onto C is one-to-one, which completes the proof.

So *uniqueness* of free algebras is established. We are going now to discuss the *existence*. If Q is inconsistent, there are no Q-V-algebras at all; if Q is semi-consistent only Q-V-algebras with one element exist which are all isomorphic and so only free Q-V-algebras with one generator exist. If Q is consistent, the notion of the quasi-free algebra can help us to investigate the existence of free algebras. We use the notation of the proof of theorem 1 for the Q-quasi-free V-algebra with n generators. Two different elements x and y of W_n cannot be equivalent, because the consistency of Q implies the existence of a Q-V-algebra with at least two elements and so a specialization exists which turns x and y into different elements. So all elements of W_n belong to different classes. We call the set of those classes W_n'; W_n' has cardinal number n. Obviously B is generated by W_n'. Every mapping of W_n into a Q-V-algebra A gives a specialization of the V-W_n-polynomials and equivalent V-W_n-polynomials give the same element. So every mapping of W_n' into A gives rise to a mapping of B into A which obviously is homomorphic. If B would be a Q-V-algebra, B would suffice all requirements of a free Q-V-algebra with n generators. So we have found:

L e m m a 1: If Q is a consistent V-W-axiom system, a free Q-V-algebra with n generators exists if and only if the Q-quasi-free V-algebra with n generators is a Q-V-algebra. If this is so the

Q-quasi-free V-algebra with n generators is a free Q-V-algebra with n generators.

We are going now to prove a theorem which gives a necessary and a theorem which gives a sufficient condition imposed on an axiom system for existence of free algebras.

We call two V-W-axiom systems Q_1 and Q_2 *equivalent* if every Q_1-V-algebra is a Q_2-V-algebra and conversely.

It is well-known from mathematical logic that to every axiom system corresponds an equivalent axiom system in which all axioms have the so-called conjunctive normal form (see e.g. Hilbert-Ackermann, Grundzüge der theoretischen Logik, 2. Auflage, p. 10). Furthermore we can split up a conjunction into separate axioms; this process transforms the axiom system into an equivalent axiom system in which all axioms have the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 1$ in which the L_i are V-W-identities or negations of V-W-identities. If Q contains such an axiom $L_1 \vee \ldots \vee L_m$ with $m > 1$ and no specialization into a Q-V-algebra exists for which L_1 turns into a true and all other L_i into false statements, the axiom system Q_1 which arises from Q by replacing $L_1 \vee \ldots \vee L_m$ by $L_2 \vee \ldots \vee L_m$, is equivalent to Q. It is clear that a Q_1-V-algebra is a Q-V-algebra. If there would be a Q-V-algebra which would not be a Q_1-V-algebra there would be a specialization into that algebra which would turn $L_1 \vee \ldots \vee L_m$ into a true and $L_2 \vee \ldots \vee L_m$ into a false statement, i.e. L_1 into a true statement and all other L_i into false statements. But such a specialization does not exist and so all Q-V-algebras are Q_1-V-algebras, thus Q and Q_1 are equivalent.

We call a V-W-axiom system Q a *reduced* V-W-axiom system, if all axioms have the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 1$ and all L_i are identities or negations of identities and if in the case that $m > 1$ for every i $(i = 1, \ldots m)$ there exists a specialization into a Q-V-algebra which turns L_i into a true and every L_j with $j \neq i$ into a false statement. We have proved:

L e m m a 2. To every V-W-axiom system there exists an equivalent reduced V-W-axiom system.

If Q is a reduced V-W-axiom system and $L_1 \vee \ldots \vee L_m$ with $m > 1$ an axiom P of Q, there exists a specialization into a Q-V-algebra for which L_1 turns into a true statement and a specialization into a Q-V-algebra for which L_1 turns into a false statement. Now assume the existence of a free Q-V-algebra F with a number of generators $\geqslant$ the number of variables occuring in L_1. Let S be the generating set of F. Choose an S-specialization such that different variables in L_1 turn into different elements of S. To every specialization into a Q-V-algebra corresponds a mapping of S into that algebra which may be extended to a homomorphism of F into that algebra. If L_1 is an identity, it must turn into a false statement by the S-specialization, for if it would turn into a true statement, the resulting equality would make that every specialization into a Q-V-algebra would turn L_1 into a true statement, which is not so. In the same manner we find that if L_1 is the negation of an identity, it must turn into a true statement. Now if $L_1, \ldots, L_m$ are all identities and a free algebra exists with a number of generators $\geqslant$ the number of variables occuring in P we get a contradiction, because we may choose an S-specialization such that different variables in P turn into different elements of S. Then all L_i turn into false statements and so F does not satisfy P. So we have found:

Theorem 2. If for a set V of operations a reduced V-W-axiom system contains an axiom P of the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 2$ and in which all L_i are identities, free Q-V-algebras with a number of generators $\geqslant$ the number of variables occuring in P do not exist. So if for a set V of operations and for a reduced V-W-axiom system free Q-V-algebras exist for every number of generators, all axioms $L_1 \vee \ldots \vee L_m$ of Q have $m = 1$ or contain at least one negation of an identity.

This theorem gives a necessary condition for the existence of free algebras; the following theorem gives a sufficient condition.

Theorem 3. If for a set V of operations a consistent V-W-axiom system Q contains only axioms of the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 1$ in which at most one of the L_i is an identity and the other L_i are negations of identities, a free Q-V-algebra exists for any number of generators.

Proof: According to lemma 1 it suffices to prove that for any cardinal number n the Q-quasi-free V-algebra B_n with n generators is a Q-V-algebra. Assume this were not so for a certain n and let W_n be the set of variables used for the definition of B_n. Then there must be a V-W-axiom $L_1 \vee \ldots \vee L_m$ of Q and a B_n-specialization such that by this specialization $L_1 \vee \ldots \vee L_m$ turns into a false statement about elements of B_n, i.e. all L_i turn into false statements about elements of B_n. Now choose representants from the classes of B_n; the two V-W-polynomials occuring in L_i turn into V-W_n-polynomials α_i and β_i, which are equivalent if L_i is the negation of an identity and not equivalent if L_i is an identity. Take first the case that all L_i are negations of identities. Because Q is consistent we can choose a Q-V-algebra A. Take an arbitrary A-specialization of W_n; α_i and β_i turn into the same element of A. Now the representants of the classes of B_n which are substituted for the variables of W in the B_n-specialization turn into elements of A by the A-specialization of W_n; so the A-specialization of W_n induces an A-specialization of W which turns the two polynomials occuring in L_i into the same element of A and so L_i into a false statement for all $i = 1, \ldots, m$. This contradicts the fact that A is a Q-V-algebra. Now take the case that one L_i is an identity and the other L_i are negations of identities. Without loss of generality we may assume L_1 is an identity and $L_2, \ldots, L_m$ are negations of identities. Take an arbitrary A-specialization of W_n into an arbitrary Q-V-algebra A. With the same argument as in the preceding case we see that this induces an A-specialization of W into A and that the L_i turn into false statements for $i = 2, \ldots, m$. Because A is a Q-V-algebra, L_1 must turn into a true statement and so α_1 and β_1 turn into the same element of A. Because this holds for all A-specializations of W_n into all Q-V-algebras A, it follows that α_1 and β_1 are equivalent. But we had already found that α_1 and β_1 are not equivalent because L_1 is an identity. This leads to a contradiction. If $m = 1$ we proceed in a similar manner. Thus theorem 3 is proved.

We remark that in theorem 3 Q needs not be reduced. This is not so strange because a not-reduced axiom system of the form required in theorem 3 remains in such a form after reduction.

We may bring the requirement imposed on the axioms in theorem 3 in the following more convenient form: the axioms of Q are of one of the following four forms: M, $\neg M$, $M_1 \wedge \ldots \wedge M_k \rightarrow M_{k+1}$, $M_1 \wedge \ldots \wedge M_k \rightarrow \neg M_{k+1}$ with $k \geqslant 1$ and in which all M and M_i are identities.

For direct products of algebras a theorem may be given which is highly analogous to theorem 3. It runs as follows:

T h e o r e m 4. If V is a set of operations and Q a reduced V-W-axiom system, a direct product of any Q-V-algebras is a Q-V-algebra if and only if all axioms of Q are of the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 1$ in which at most one of the L_i is an identity and all other L_i are negations of identities [1]).

1) After finishing my paper I discovered that this theorem is implicitly contained in a paper of J. C. C. Mc Kinsey (J. Symbolic Logic **8** (1943), 61–76). He does not state the theorem explicitly and his assumptions and proof are somewhat (but not essentially) different from mine. For this

Proof: Two elements of a direct product are equal if and only if all corresponding "components" are equal. From this we infer that if the axioms of Q have the prescribed form, a direct product of Q-V-algebras is a Q-V-algebra. A specialization into the direct product induces a specialization into all factors and conversely. If for one factor a negation of an identity occuring in an axiom turns into a true statement, i.e. an inequality, the same inequality holds for the direct product. If an axiom contains an identity and for all factors this identity turns into a true statement, i.e. an equality, the same equality holds for the direct product. In both cases the specialization into the direct product turns the axiom into a true statement. Now assume that Q contains an axiom with at least two identities. Let the axiom be $M_1 \vee M_2 \vee \ldots$ in which M_1 and M_2 are identities, possibly followed by some more identities or negations of identities. As Q is a reduced axiom system, a Q-V-algebra A_1 and an A_1-specialization exist for which M_1 turns into a true statement, i.e. an equality and for which all other M_i turn into false statements and a Q-V-algebra A_2 and an A_2-specialization exist for which M_2 turns into a true statement, i.e. an equality and for which all other M_i turn into false statements. The corresponding specialization of the direct product of A_1 and A_2 turns all M_i into false statements: for the identities M_1 and M_2 this is so because they turn into false statements in one of the two factors, for the remaining M_i this is so because they turn into false statements for both factors. Thus the direct product does not satisfy the axiom. This completes the proof of theorem 4.

We conclude this paper with a remark on the possibility of extending the admissible form of the axioms in the discussion of this paper. From the standpoint of first-level predicate calculus our axioms are such that all variables are bound by universal quantifiers $(\forall x)$. If we admit also the occurence of existential quantifiers $(\exists x)$, we may eliminate those existential quantifiers by introducing for every existential quantifier a new operation which depends on all variables which are bound by the quantifiers which stand before the existential quantifier. If there are no quantifiers before the existential quantifier we can avoid the introduction of a zero-ary operation in taking a unary operation $O\ (x)$ and adding the axiom $(\forall x)\ (\forall y)\ (O\ (x) = O\ (y))$. So e.g. if we require for a multiplicative system (algebra with one binary operation) that it has an identity element, which we can express in the axiom $(\exists\ x)$

reason and because his formulation is rather different from mine, I decided to maintain the theorem in my paper.

$(\forall y)\ (xy = y) \wedge (yx = y)$, we may also introduce the new operation $E(x)$ and the axioms $(\forall x)\ (\forall y)\ (E(x) = E(y))$ and $(\forall x)$ $(\forall y)\ (E(x)\, y = y) \wedge (yE(x) = y)$. However we must bear in mind, that the introduction of new operations may alter our algebraic notions such as subalgebra, subalgebra generated by a subset, homomorphism and thus also free algebra. This shows that the application of our results to the extended class of axioms is not immediate. Some examples will illustrate this.

We may define a group as a multiplicative system which satisfies certain axioms. A homomorphism is a mapping such that if x is mapped into x' and y into y', xy is mapped into $x'y'$. We can avoid the occurence of existential quantifiers in the axioms by introducing a new unary operation x^{-1}, called inverse. The following axioms determine the groups: $(xy)\, z = x(yz)$, $xx^{-1} = yy^{-1}$, $x\, (xx^{-1}) = x$. For the notion of homomorphism the introduction of the inverse-operation is harmless, because a homomorphism as defined above is also a homomorphism for the inverse-operation. The notion subalgebra generated by a subset is altered: a cyclic group consisting of the powers with integral exponents of an element a is generated by a in the new sense, but is not generated by a as a multiplicative system. However the new interpretation is the more natural and we shall adopt it for the definition of a free group. We may now apply theorem 3 and conclude that free groups exist for all numbers of generators.

An analogous discussion holds for fields. Beside the operations of addition, subtraction and multiplication we have to introduce a unary operation corresponding to the inverse. For the zero-element this operation could be arbitrary. In order to ensure that the introduction of the new operations leaves unaltered the concepts of homomorphism and of subalgebra generated by a subset, we postulate $0^{-1} = 0$. So the axioms for the inverse-operation read: $(x - x)^{-1} = x - x$, $(x+x = x) \vee (y+y = y) \vee (xx^{-1} = yy^{-1})$ and $(x + x = x) \vee (x\,(xx^{-1}) = x)$. These axioms are easily shown to be reduced. Applying theorem 2 we conclude that for all numbers of generators free fields do not exist.

Amsterdam, Mathematisch Centrum.

BOEKBESPREKINGEN.

W. C. COEPIJN, *Berekening van Staafwerken volgens de Methode ,,Cross''*. Tweede druk P. Noordhoff N.V. — Groningen-Batavia, 104 blz., druk (1949), 152 fig., 8 graphieken. *f* 2,35.

Het berekenen van hyperstatistische systemen d.m.v. de vereffeningsmethode van CROSS, heeft overal veel bijval genoten, dank zij de eenvoud der methode, de overzichtelijkheid en de geringe kans op het maken van rekenfouten.

Bedoeling van de schrijver is de methode ingang te doen vinden in het middelbaar onderwijs, wat zeer goed mogelijk is, aangezien voor het begrijpen en toepassen ervan, de kennis van slechts enkele klassieke belastingsgevallen van balken vereist is.

In het boek beperkt de auteur zich tot de vlakke systemen (balken en ramen) en vaste lasten.

In het eerste hoofdstuk krijgen wij een overzicht van de statische en hyperstatische stelsels, vervolgens wordt tot de balkentheorie, voorzover nodig tot het begrijpen van de Cross-methode, overgegaan, dan volgt de berekening van doorlopende balken en ramen. Ten slotte behandelt schrijver de balk met veranderlijk traagheidsmoment, dat echter beperkt blijft tot balken met lineair veranderlijke hoogte.

De theoretische uiteenzetting wordt duidelijk gemaakt door talrijke rekenvoorbeelden. — Druk en figuren zijn goed verzorgd.

R. VANCROMBUGGE

HAROLD JEFFREYS and BERTHA SWIRLES JEFFREYS, *Methods of Mathematical Physics*. Second Edition. Cambridge University Press, 1950. XII + 708 pp. £ 4. 4 s.

The fact that a second edition of this book became necessary four years after the first one, is the best proof that this excellent book was a success.

Referring only to the title, one should expect, that this book has much in common with Courant and Hilbert's "Methoden der Mathematischen Physik". This, however, is not true; whereas the latter emphasizes more the solutions of eigenvalue-problems and makes a great use of variational methods, the former provides "an account of those parts of pure mathematics that are most frequently needed in physics" and gives from the beginning numerous worked-out examples, taken from mechanics, elasticity, geophysics, hydrodynamics, the theory of heat, electricity and quantum-mechanics. So the book can already be read with great profit by students in mathematics, physics or engineering having finished their ,,candidature'' (first two years of universitystudy in Belgium).

Some chapters of the book treat the operational calculus and in them most of the text of Prof. H. JEFFREYS' "Operational Methods in Mathematical Physics" has been incorporated.

Besides the correction of errata considerable revisions have been made in the second edition of this book: some methods have been extended, the chapter on multiple integrals has been almost completely rewritten, correc-